Indian Statistical Institute, Bangalore
B. Math.
First Year, First Semester
Analysis I

Back paper Examination — Date : Dec. 30, 2019
Total Marks: 105 Maximum marks: 100 — Time: 3 hours
Teacher: B V Rajarama Bhat

Notation: $\mathbb{N} = \{1, 2, 3, \ldots\}$ is the set of natural numbers and $\mathbb{R}$ is the set of real numbers.

(1) Let A be the set of all finite subsets of rational numbers. Is A countable or uncountable? Prove your claim. [15]

(2) Show that every sequence of real numbers has a monotonic subsequence. [15]

(3) Compute lim inf and lim sup of following sequences:
(i) $a_n = \frac{1}{2^n} + (-1)^n \frac{1}{3n},\ n \geq 1$;
(ii) $b_n = |(\frac{1}{2} - \frac{1}{n})| + 5,\ n \geq 1$;
(iii) $c_n = a_n + b_n,\ n \geq 1$.
[15]

(4) Find all continuous functions $g : \mathbb{R} \to \mathbb{R}$ which satisfy

$$g(x + \frac{1}{n}) = g(x),\ \forall n \in \mathbb{N} \text{ and } x \in \mathbb{R}.$$

[15]

(5) Let a, b, a_1, b_1 be real numbers with $a < b$ and $a_1 < b_1$. Let $f : [a, b] \to [a_1, b_1]$ be a continuous bijection. Show that f^{-1} is continuous. Suppose $a < c < b$ and $c_1 = f(c)$. Show that if f is differentiable at c and $f'(c) \neq 0$ then f^{-1} is differentiable at c_1. [15]

(6) State and prove mean value theorem for a real valued function on an interval $[a, b]$. [15]

(7) Find first three terms of the Taylor expansion of the function $g : [-2, 2] \to \mathbb{R}$ defined by

$$g(x) = \frac{2x}{(x-3)(x+5)},\ x \in [-2, 2]$$

around the point $x_0 = 0$ and $x_1 = 1$. [15]